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Approximation in Thin Film Brick-Wall Model

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Abstract The thin film brick-wall model is an important method on calculating the entropy of the non-static black hole. But we can see in this paper that the approximation used in the thin film brick-wall model is a rate of two infinitesimal parameters in fact. in this paper, we make an exact calculation of the approximation term and give the solution in the Schwarzschild black hole space-time.

Keywords Entropy · Thin film brick-wall model · Cut-off factor

Since Bekenstein and Hawking [1–3] had proved the entropy of a black hole is proportional to its surface area, many works have been done to study the black hole entropy. An important method is the brick-wall method put forward by 't Hooft [4] who assumed the black hole's entropy was identified by a thermal gas outside the event horizon. Recently the improved thin film brick-wall model [5] which can give a clear explanation on black hole's entropy was proposed. It put forward that the entropy of a black hole mainly comes from a thin layer near the horizon. This method has been applied to various black hole models [6, 7], and the result is satisfactory.

In these papers, an approximate method was used in the process of using thin film brick-wall model to get the free energy and the entropy. In this paper, after an exact calculation of the approximation term in the Schwarzschild black hole space-time, we can find that the approximation is a rate of two infinitesimal parameters in fact. We give the way to resolve this problem at last.

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The line element of the Schwarzschild black hole is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{1}$$

where m is the mass of the black hole, the event horizon is

$$r = 2M \tag{2}$$

the temperature of the horizon is

$$T = \frac{1}{\beta} = \frac{1}{8\pi M}. \tag{3}$$

According to Klein–Gordon equation for scalar field

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)\phi = m^2\phi \tag{4}$$

as

$$\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial x^\mu}\left(r^2 \sin\theta g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)\phi - m^2\phi = 0. \tag{5}$$

Due to spherical symmetry Schwarzschild black hole, we can separate the variable

$$\phi(r,\theta,\varphi,t) = f(r)Y_{lm}(\theta,\varphi)e^{-iEt} \tag{6}$$

the equation of radial part can be written as

$$\left(1 - \frac{2M}{r}\right)^{-1} E^2 f(r) + \frac{1}{r}\frac{d}{dr}\left[r^2\left(1 - \frac{2M}{r}\right)\frac{d}{dr}f(r)\right] - \left(m^2 + \frac{l(l+1)}{r^2}\right)f(r) = 0. \tag{7}$$

Rewriting $f(r)$ as $f(r) = \exp(iS(r))$, we can obtain the following equation by the WKB (Wenzel, Kramers, Brillouin) approximation

$$k_r^2 = \left(\frac{\partial S(r)}{\partial r}\right)^2 = \left[\left(1 - \frac{2M}{r}\right)^{-1} E^2 - \left(m^2 + \frac{l(l+1)}{r^2}\right)\right]\left(1 - \frac{2M}{r}\right)^{-1} \tag{8}$$

where k_r is wavenumber.

According to the theory of canonical ensemble, the free energy can be written as

$$\beta F = \sum_E \ln(1 - e^{-\beta E}). \tag{9}$$

In term of semiclassical theory, assuming the energy E is continuous, we can replace the sum by integration

$$\begin{aligned}\beta F &= \int_0^{+\infty} dE g(E)\ln(1-e^{-\beta E}) = \int_0^{+\infty} d\Gamma(E)\ln(1-e^{-\beta E})\\&= \Gamma(E)\ln(1-e^{-\beta E})|_0^{+\infty} - \int_0^{+\infty}\frac{\Gamma(E)e^{-\beta E}}{1-e^{-\beta E}}\beta dE\\&= -\beta\int_0^{+\infty}\frac{\Gamma(E)}{e^{\beta E}-1}dE\end{aligned}$$

where $\Gamma(E)$ is the total number of microscopic states with energy less than E, $g(E) = d\Gamma/dE$ is the density of states, we have integrated by parts in the third step of the above equation.

According to semiclassical quantum theory, we have

$$\begin{aligned}\Gamma(E) &= \sum_{l,m} n_r(E,l,m) = \sum_l (2l+1)n_r(E,l)\\&= \int_l (2l+1)dl\frac{1}{\pi}\int_r k_r(r,E,l)dr.\end{aligned}$$

Therefore, the express of free energy (9) can be written as

$$F = -\frac{1}{\pi}\int_0^{+\infty} dE\int_r dr\int_l (2l+1)dl\frac{k_r(r,E,l)}{e^{\beta E}-1}. \tag{10}$$

With regard to (8), we can get

$$\begin{aligned}F = -\frac{1}{\pi}\int_0^{+\infty} dE\int_r dr\int_l (2l+1)dl(e^{\beta E}-1)^{-1}\Bigg\{\Bigg[\left(1-\frac{2M}{r}\right)^{-1}E^2\\ -\left(m^2+\frac{l(l+1)}{r^2}\right)\Bigg]\left(1-\frac{2M}{r}\right)^{-1}\Bigg\}^{1/2}.\end{aligned} \tag{11}$$

The integration of (11) is very simple. The upper limit of the integration with respect to l is taken so that k_r^2 is non-negative, and the lower limit is zero. After the integration, we get

$$F = -\frac{2}{3\pi}\int_0^{+\infty}\frac{dE}{e^{\beta E}-1}\int_r dr\frac{r^2}{(1-2M/r)^2}\left[E^2-\left(1-\frac{2M}{r}\right)m^2\right]^{3/2}. \tag{12}$$

The integration with respect to E and r is very complicated. According to thin film brick-wall model, we take only the free energy of a thin layer near horizon of a black hole, the integration with respect to r must be limited in region $r_h+\varepsilon \le r \le r_h+\varepsilon+\delta$, where r_h is the black hole horizon, both ε and δ are the positive infinitesimal parameters, and the coefficient before m^2 is naturally zero. So in term of (12), we have

$$F = -\frac{2}{3\pi}\int_0^{+\infty}\frac{E^3dE}{e^{\beta E}-1}\int_{r_h+\varepsilon}^{r_h+\varepsilon+\delta} r^2\left(1-\frac{2M}{r}\right)^{-2}dr. \tag{13}$$

In (13), the integration with respect to E is very simple, we can give the result $\pi^4/(15\beta^4)$ at once. In Ref. [6] the last part of the integration is simply approximated and get

$$F = -\frac{2}{3\pi}\int_0^{+\infty}\frac{E^3 dE}{e^{\beta E}-1}\int_{r_h+\varepsilon}^{r_h+\varepsilon+\delta} r^2\left(1-\frac{2M}{r}\right)^{-2} dr$$
$$= \frac{2}{3\pi}\frac{\pi^4}{15\beta^4}(2M)^4\left(\frac{1}{\varepsilon+\delta}-\frac{1}{\varepsilon}\right)$$
$$= -\frac{32\pi^3 M^4}{45\beta^4}\frac{\delta}{\varepsilon(\varepsilon+\delta)} \tag{14}$$

and we can get the entropy

$$S = \beta^2\frac{\partial F}{\partial\beta} = \frac{128\pi^3 M^4}{45\beta^3}\frac{\delta}{\varepsilon(\varepsilon+\delta)} = \frac{1}{4}\frac{A_h}{90\beta}\frac{\delta}{\varepsilon(\varepsilon+\delta)} \tag{15}$$

where A_h is the area of the black hole horizon. Compare the result with B-H entropy, we have

$$\frac{\delta}{\varepsilon(\varepsilon+\delta)} = 90\beta = \frac{90}{T}. \tag{16}$$

If we make an exact calculation of the integration of (13),

$$F = -\frac{2}{3\pi}\int_0^{+\infty}\frac{E^3 dE}{e^{\beta E}-1}\int_{r_h+\varepsilon}^{r_h+\varepsilon+\delta} r^2\left(1-\frac{2M}{r}\right)^{-2} dr$$
$$= \frac{2}{3\pi}\frac{\pi^4}{15\beta^4}\int_{r_h+\varepsilon}^{r_h+\varepsilon+\delta}\Bigg[(r-2M)^2 + 8M(r-2M)$$
$$+ 24M^2 + \frac{32M^3}{r-2M} + \frac{(2M)^4}{(r-2M)^2}\Bigg]dr$$
$$= -\frac{32\pi^3}{45\beta^4}\left[32M^3\ln\frac{\varepsilon+\delta}{\varepsilon} - (2M)^4\left(\frac{1}{\varepsilon+\delta}-\frac{1}{\varepsilon}\right)\right] \tag{17}$$

we can see that the first term and the second term in step 2 is naturally zero, and there is a term $\ln((\varepsilon+\delta)/\varepsilon)$ in (17) which does not appear in (12). Because ε and δ are two infinitesimal parameters, the result of the term is an unknown value. It can not be simply omitted.

In fact, only after calculating the entropy and comparing it with S-H entropy, according to (16)

$$\frac{\delta}{\varepsilon(\varepsilon+\delta)} = 90\beta = \frac{90}{T} \tag{18}$$

as

$$\frac{\delta}{\varepsilon} = \frac{1}{\frac{T}{90\varepsilon}-1} \tag{19}$$

together with (15), we can obtain

$$\ln\frac{\varepsilon+\delta}{\varepsilon} = \ln\left(1+\frac{\delta}{\varepsilon}\right) = \ln\left(1+\frac{1}{\frac{T}{90\varepsilon}-1}\right) = 0. \tag{20}$$

The final result is the same as (14).

In summary, using the thin film brick-wall model in the simplest Schwarzschild black hole space-time, we have made an exact calculation of the approximate term. We find that the term is the rate of two infinitesimal parameters, but we can deal with the problem by using the cut-off factor gotten later.

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